COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 15 (3) (1995) 373–377

NOTE ON A CONJECTURE OF TOFT

T. R. JENSEN and F. B. SHEPHERD

Received September 16, 1993
Revised February 21, 1994

A conjecture of Toft [17] asserts that any 4-critical graph (or equivalently, every 4-chromatic graph) contains a fully odd subdivision of K_4. We show that if a graph G has a degree three node v such that $G-v$ is 3-colourable, then either G is 3-colourable or it contains a fully odd K_4. This resolves Toft's conjecture in the special case where a 4-critical graph has a degree three node, which is in turn used to prove the conjecture for line-graphs. The proof is constructive and yields a polynomial algorithm which given a 3-*degenerate* graph either finds a 3-colouring or exhibits a subgraph that is a fully odd subdivision of K_4. (A graph is 3-degenerate if every subgraph has some node of degree at most three.)

A *subdivision* H of K_n is any graph obtained by replacing some edges uv by a uv path (whose internal nodes are then of degree two in H). The subdivision is called *fully odd* if each of the added paths has an odd number of edges. An *odd* K_n is a subdivision where each of the triangles becomes an odd cycle; thus each fully odd K_n is an odd K_n but the converse is not true.

Hajós [10] conjectured that if a graph contains no subgraph which is the subdivision of a K_n, then it can be $(n-1)$-coloured. This was proved for $n \leq 4$ by Dirac [3]*, disproved for $n \geq 7$ by Catlin [1], and remains open for $n=5,6$. Toft [17] conjectured a stronger version of Dirac's result, that if a graph has no subgraph which is a fully odd K_4, then it is 3-colourable. Paul Seymour drew attention to this conjecture at the Graph Minors Workshop held in Seattle in 1991, see [18]. An intermediate result due to Catlin states that if a graph has no odd K_4 as a subgraph, then it is 3-colourable. The class of graphs with no odd K_4 has been further studied in [8] where a recognition algorithm is given which is based on a decomposition theorem. This is used in [9] to show that these graphs are t-perfect. This stronger property does not hold for graphs with no fully odd K_4, since subdividing once each of the edges incident to a specified node in K_4 results in a graph which is not t-perfect and has no fully odd K_4. A related result due to Sewell and Trotter [14] confirms a conjecture of Chvátal [2] which asserts that any α-*critical* graph (i.e., deletion of any edge increases the size α of a maximum stable set) contains a fully odd K_4 as a subgraph.

Mathematics Subject Classification (1991): 05 C 15, 05 C 38, 05 C 85

* B. Toft has recently pointed out that a proof of Hajós' conjecture for $n \leq 4$ was also contained in a classical paper of H. Hadwiger in 1943.

In the present paper we give a sufficient condition for a graph to contain a fully odd K_4. As a corollary, it is shown that if G is 4-chromatic but possesses a degree three node z such that $G-z$ is 3-colourable, then G contains a fully odd K_4.

Theorem 1. *Let G be a graph with a node $z \in V(G)$ and suppose that there is a colouring c of $G-\{z\}$ such that z has neighbours z_1, z_2, z_3 in G where $c(z_i)=i$ and for each $i=1,2,3$, there is a z_i, z_{i+1} path whose nodes alternate between nodes of colours i and $i+1$ (subscripts taken cyclically). Then G contains a fully odd K_4.*

Proof. We prove the result by finding either a fully odd K_4 directly, or a smaller subgraph satisfying the hypothesis.

For each $i=1,2,3$, let $P_{i,i+1}$ be a path alternating between colours i and $i+1$ (subscript arithmetic is done cyclically) whose endpoints are z_i and z_{i+1}. Clearly we can assume that G has no nodes or edges other than z, the edges zz_i, $i=1,2,3$, and the nodes and edges of the paths $P_{i,i+1}$, $i=1,2,3$.

First suppose that G has a node u of degree $d(u) \leq 2$. Then u is different from z, z_1, z_2, z_3 and $d(u)=2$. Let u have neighbours x and y, and H be obtained from G by deleting u and identifying x and y. H is a smaller graph satisfying the same hypothesis as G. Moreover, a fully odd K_4 in H is easily transformed into a fully odd K_4 in G as well. So we now assume that G has no node of degree ≤ 2.

Trivially a fully odd K_4 exists if each path $P_{i,i+1}$, $i=1,2,3$, is a single edge. So we assume that $P_{3,1}$ say, is not a single edge. Let w_1 and w_3 be the nodes on $P_{1,2}$ and $P_{2,3}$ respectively that are adjacent to z_2. Possibly $w_1=z_1$ or $w_3=z_3$, but not both equalities can hold, since $P_{3,1}$ has no nodes of degree 2. We may assume $w_1 \neq z_1$. Note that both w_1 and w_3 lie on $P_{3,1}$ since their degrees are at least 3. Define a new colouring such that z gets coloured 2, and all other colours remain the same. This defines a proper colouring of the graph $G-z_2$ and the neighbours w_1, z, w_3 of z_2 in G are coloured 1,2 and 3 respectively. Define paths $Q_{1,2}, Q_{2,3}, Q_{3,1}$ as follows. Let $Q_{1,2}$ be $P_{1,2}-z_2w_1+zz_1$, $Q_{2,3}$ be $P_{2,3}-z_2w_3+zz_3$ and $Q_{3,1}$ be the segment of $P_{3,1}$ between w_1 and w_3. We observe that none of the paths $Q_{i,i+1}$, $i=1,2,3$, contains the edge e of $P_{3,1}$ incident to z_1 in G. Hence we have a proper subgraph $G-e$ of G that satisfies the original hypothesis with z_2 in place of z, nodes w_1, z, w_3 in place of z_1, z_2, z_3 and with paths $Q_{i,i+1}$ replacing paths $P_{i,i+1}$, $i=1,2,3$. ∎

We now verify the conjecture of Toft for 4-critical graphs with a degree three node. This class includes a large proportion of the known examples of critical graphs.

Corollary 2. *If G is 4-critical and contains a node of degree three, then it contains a fully odd K_4.*

Proof. Suppose z has degree three and let $c: V \to \{1,2,3\}$ be a proper 3-colouring of $G-z$. Then clearly there is one node of each colour in $N(z)$ and for any $u, v \in N(z)$, there is a $c(u)c(v)$ path joining u and v. Otherwise we could recolour $G-z$ so that u gets colour $c(v)$ without affecting the colours of the other neighbours of z, and finally 3-colour G by colouring z with colour $c(u)$. ∎

We verify Toft's Conjecture also for line-graphs.

Corollary 3. *Every 4-chromatic line-graph contains a fully odd K_4.*

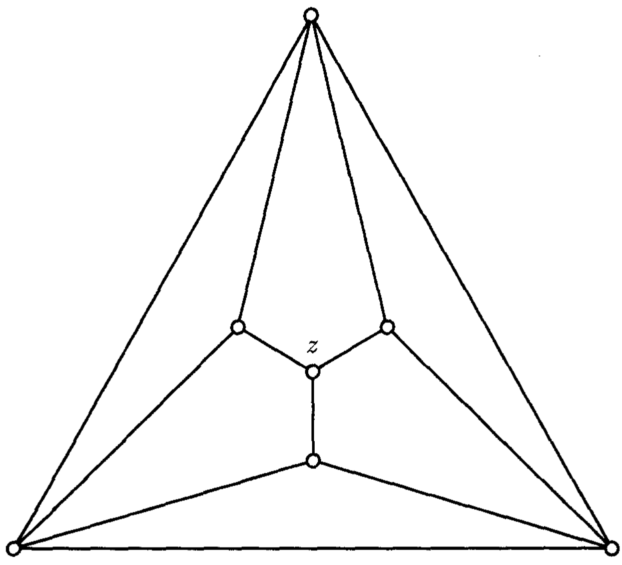

Fig. 1. The example of Figure 1 shows that the fully odd K_4 whose existence is asserted by the previous result cannot be guaranteed to contain the special node z as a degree three node.

Proof. Let H be a 4-edge-chromatic graph. We can assume that H is critical, that is, the deletion of any edge of H lowers the edge-chromatic number; equivalently, the line-graph $L(H)$ is vertex-critical. If H contains a node of degree four, then $L(H)$ contains K_4 as a subgraph and we are done. Thus H has maximum degree 3 and hence has minimum degree at least 2. If H contains a node of degree two, then $L(H)$ contains a node of a degree three and therefore a fully odd K_4 by Corollary 2.

Finally, it is well-known that H cannot be 3-regular (e.g., see [5]). One way to see this, is to let $e=uv$ be any edge of H and c be a 3-edge-colouring of $H-e$. For each $i=1,2,3$, let n_i be the number of vertices incident to an edge with colour i. Thus each n_i is even and they sum to $|V(G)|-2$. It follows that two of these numbers are $|V(G)|$ and the third is $|V(G)|-2$. Hence the same colour is missing at x and at y and so the colouring can be extended to H, a contradiction. ∎

The proof of Theorem 1 also gives an iterative procedure for finding either a 3-colouring or a fully odd subdivision of K_4 in a 3-degenerate graph (G is 3-*degenerate* if every nonnull subgraph of G has a node of degree at most three).

Let $n=|V(G)|$ and $G_n=G$. For $i=n,n-1,\dots,1$ let x_i be a node of degree at most three in G_i, and let $G_{i-1}=G_i-x_i$. The approach consists in repeatedly attempting to extend a 3-colouring of G_{i-1} to G_i, $i=1,\dots,n$. If this cannot be done immediately, then the neighbours of x_i in G_i have received three different colours when colouring G_{i-1}. An extended colouring can then be achieved by a recolouring procedure as in the proof of Corollary 2, or the conditions of Theorem 1 hold with $z=x_i$. In the latter case, the proof of the theorem then describes a polynomial-time procedure for finding a fully odd subdivision of K_4.

We note that 3-colouring is $\mathcal{NP}$-complete for the class of 3-degenerate graphs even when restricting to the planar case. This follows from the reductions of Garey, Johnson and Stockmeyer [7].

Corollary 2 shows that a minimal counterexample to Toft's conjecture is a 4-critical graph of minimum degree at least four. We do not know of any results that indicate whether this class of 4-critical graphs is relatively abundant or not, but we suspect that Corollary 2 covers most cases. Independent constructions by Simonovits [15] and Toft [16] show that there exist families of 4-critical graphs on n nodes and minimum degree $cn^{1/3}$ for $c > 0$ and infinitely many n. For these graphs there are however some difficulties involved in checking Toft's conjecture, since the constructions are not entirely explicit but are partially based on counting arguments. Gallai [6] constructed an infinite family consisting of 4-critical 4-regular graphs which can be easily checked to satisfy Toft's conjecture, and so can 4-critical 4-regular examples found by Jensen and Royle [11] and Youngs [19], as well as the planar examples by Koester [12, 13].

Finally we may illustrate how little is known about the possible degree sequences of 4-critical graphs by referring to Erdös [4] who noted that there seems to be no known examples of r-regular 4-critical graphs for any $r \geq 6$ (in fact, we do not know examples of 5-regular 4-critical graphs) but he conjectured that they exist for all values of $r \geq 6$.

Acknowledgements. The authors would like to thank one referee for particularly helpful comments and the Institute for Discrete Mathematics in Bonn for its hospitality. The second author was supported by the Von Humboldt Stiftung.

References

[1] P. A. Catlin: Hajós graph-colouring conjecture: variations and counterexamples, *J. Comb. Theory (B)* **26**, (1979), 268–274.

[2] V. Chvátal: On certain polytopes associated with graphs, *J. Comb. Theory (B)* **18**, (1975), 138–154.

[3] G. A. Dirac: A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27**, (1952), 85–92.

[4] P. Erdös: On some aspects of my work with Gabriel Dirac, in *Graph Theory in Memory of G. A. Dirac. Ann. of Discrete Math.* **41**, North-Holland 1989, 111–116.

[5] S. Fiorini, R. J. Wilson: *Edge-colourings of graphs*, Pitman 1977.

[6] T. Gallai: Kritische Graphen I, *Publ. Math. Inst. Hungar. Acad. Sci.* **8**, (1963), 165–192.

[7] M. R. Garey, D. S. Johnson, L. Stockmeyer: Some simplified $\mathcal{NP}$-complete graph problems, *Theor. Comp. Sci.* **1**, (1976), 237–267.

[8] A. M. H. Gerards, L. Lovász, A. Schrijver, Shih, P. D. Seymour, K. Truemper: *manuscript*.

[9] A. M. H. Gerards, A. Schrijver: Matrices with the Edmonds-Johnson property, *Combinatorica* **6**, (1986), 403–417.

[10] G. Hajós: Über eine Konstruktion nicht n-färbbarer Graphen, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **10**, (1961), 116-117.

[11] T. R. Jensen, G. F. Royle: Small graphs of chromatic number 5 – a computer search, *J. Graph Theory*, **19**, (1995), 107–116.

[12] G. KOESTER: Note on a problem of T. Gallai and G. A. Dirac, *Combinatorica* **5**, (1985), 227–228.

[13] G. KOESTER: 4–critical 4–valent planar graphs constructed with crowns, *Math. Scand.* **67**, (1990), 15–22.

[14] E. C. SEWELL, L. E. TROTTER, JR.: Stability critical graphs and even subdivisions of K_4, *submitted to J. Comb. Theory (B)*.

[15] M. SIMONOVITS: On colour-critical graphs, *Studia Sci. Math. Hungar.* **7**, (1972), 67–81.

[16] B. TOFT: Two theorems on critical 4-chromatic graphs, *Studia Sci. Math. Hungar.* **7**, (1972), 83–89.

[17] B. TOFT: Problem 11, in *Recent Advances in Graph Theory*, Academia Praha 1975, 543–544.

[18] D. B. WEST: Open problems, *SIAM J. Discrete Math. Newsletter*, **2(1)**, (1991), 10–12.

[19] D. A. YOUNGS: Gallai's problem on Dirac's construction, *Discrete Math.* **101**, (1992), 343–350.

T. R. Jensen

Department of Mathematics
and Computer Science
Odense University
Odense, Denmark
trjensen@imada.ou.dk

F. B. Shepherd

Mathematics and Operational Research
London School of Economics
London, U.K.
shepherf@vax.lse.ac.uk